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PRIME IDEALS IN RESTRICTED DIFFERENTIAL OPERATOR RINGS

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ABSTRACT

In this paper restricted differential operator rings are studied. A restricted differential operator ring is an extension of a k-algebra R by the restricted enveloping algebra of a restricted Lie algebra $\mathfrak{g}$ which acts on R. This is an example of a smash product $R \# H$ where $H = u(\mathfrak{g})$. We actually deal with a more general twisted construction denoted by $R * \mathfrak{g}$ where the restricted Lie algebra $\mathfrak{g}$ is not necessarily embedded isomorphically in $R * \mathfrak{g}$. Assume that $\mathfrak{g}$ is finite dimensional abelian. The principal result obtained is Incomparability, which states that prime ideals $P_1 \subsetneq P_2 \subset R * \mathfrak{g}$ have different intersections with R. We also study minimal prime ideals of $R * \mathfrak{g}$ when R is $\mathfrak{g}$-prime, showing that the minimal primes are precisely those having trivial intersection with R, that these primes are finite in number, and their intersection is a nilpotent ideal. Prime and primitive ranks are considered as an application of the foregoing results.

Introduction

The study of prime ideals in ring extensions such as differential operator rings and crossed products was undertaken in [4, 12]. Prime ideals have been studied for finite normalizing extensions, extending results for crossed products of finite groups, in [5, 6, 7]. More recently, Hopf algebra smash products have been studied in [2, 3], extending results on group actions.

In this paper restricted differential operator rings are studied. A restricted differential operator ring is an extension of a k-algebra R by the restricted enveloping algebra of a restricted Lie algebra $\mathfrak{g}$ which acts on R. This is an example of a smash product $R \# H$ where $H = u(\mathfrak{g})$. We actually deal with a more general twisted construction denoted by $R * \mathfrak{g}$ where the restricted Lie algebra $\mathfrak{g}$ is not necessarily embedded isomorphically in $R * \mathfrak{g}$.

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Assume that $\mathfrak{g}$ is finite dimensional abelian. The principal result obtained is Incomparability (Theorem 19) which states that prime ideals $P_1 \subsetneq P_2 \subset R * \mathfrak{g}$ have different intersections with R. We also study minimal prime ideal of $R * \mathfrak{g}$ when R is $\mathfrak{g}$-prime, showing that the minimal primes are precisely those having trivial intersection with R, that these primes are finite in number, and their intersection is a nilpotent ideal. This characterization is stated in Theorem 22. Prime and primitive ranks are considered in Theorem 24 as an application of the foregoing results.

In studying prime ideals in $R * \mathfrak{g}$ we are led to the study of $\mathfrak{g}$-prime rings R. Here it is useful to construct the symmetric quotient ring S of R and the extension $S * \mathfrak{g}$. Enough of the prime ideal structure is preserved by $S * \mathfrak{g}$ so we may lift certain questions about primes. The strategy followed is similar to that of [4].

The paper begins with a construction of $R * \mathfrak{g}$ and its basic properties. Lemmas 2 through 10 deal with $\mathfrak{g}$-prime ideals, the quotient rings S and the extensions $S * \mathfrak{g}$. A partial ideal correspondence is stated in Lemma 10, and this fact is used heavily to transfer questions to $S * \mathfrak{g}$. The proof of Incomparability breaks down into two cases: When $\mathfrak{g}$ is outer on S, and $\mathfrak{g}$ is abelian, $R * \mathfrak{g}$ has the *ideal intersection property* where every nonzero ideal of $R * \mathfrak{g}$ has nonzero intersection with R. Theorem 11 and its corollaries deal with the outer case. When $\mathfrak{g}$ is inner in S, Theorem 17 reduces the question to prime ideals of a finite dimensional algebra over a field. Incomparability is proved in Theorem 19 and is then applied to complete the correspondence started in Lemma 10. More precisely, theorem 20 states that the prime ideals of $R * \mathfrak{g}$ having zero intersection with R is in bijective correspondence with the similarly defined set of primes in $S * \mathfrak{g}$. This is then used to characterize minimal primes in Theorem 21 and our "Going Down" result, Corollary 22. Primitive ideals are examined in Lemma 23, where versions of classical restriction and induction are stated. This result allows us to handle primitive ideals and compute the prime and primitive ranks of $R * \mathfrak{g}$, resulting in Theorem 24.

In closing we present an example showing that restricted differential operator rings are not (intermediate) normalizing extensions and therefore are not covered by results in [5, 6, 7]. It is also remarked that Theorem 17 yields information on smash products $R \# H$ where the action of the Hopf algebra H is *inner* on R.

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NOTATION. Throughout this chapter R shall denote a k-algebra of characteristic $p > 0$ where k is contained in R. $\mathfrak{g}$ shall denote a restricted Lie algebra over k which acts on R via a k-linear map $\delta : \mathfrak{g} \to \mathrm{Der}_k R$. δ need not be a restricted Lie algebra homomorphism, but is twisted in a way (as we shall see) so that we may construct an associative restricted differential operator ring with underlying space $R \otimes u(\mathfrak{g})$.

If δ happens to be a homomorphism, we obtain the smash product $R \# u(\mathfrak{g})$ (see [2, 3]).

Basic information on restricted Lie algebras is contained in [10].

Let $x_1, \ldots, x_n$ be a basis for $\mathfrak{g}$. Then a version of the Poincare–Birkhoff–Witt theorem says that monomials of the form $x_1^{\nu_1} \cdots x_n^{\nu_n}$, where $0 \leqq \nu < p$, from a basis for $u(\mathfrak{g})$. Such monomials are said to be *standard* and are denoted by x^ν where $\nu = (\nu_1, \ldots, \nu_n)$. Set $|\nu| = \Sigma \nu_i$, $\delta^\nu = \delta_{x_1}^{\nu_1} \cdots \delta_{x_n}^{\nu_n}$ and $\delta_{x_i} = \delta_i$.

Restricted differential operator rings

Let us discuss how the associative algebra $R * \mathfrak{g}$ might be constructed. We begin by constructing a twisted semidirect product of R, viewed as a restricted Lie algebra, and $\mathfrak{g}$ which acts on R via the map $\delta : \mathfrak{g} \to \mathrm{Der}_k R$. To this end, let $R \oplus \mathfrak{g}$ be the direct sum of R and $\mathfrak{g}$ and define the commutator and pth power map in $R \oplus \mathfrak{g}$ by:

$$[(a, x), (b, y)] = ([a, b] + \delta_x(b) - \delta_y(a) + t(x, y), [x, y]),$$

and

$$(a, x)^{[p]} = (a^p + \pi(a, x), x^{[p]}),$$

where $a, b \in R$ and $x, y \in \mathfrak{g}$. Here $t : \mathfrak{g} \times \mathfrak{g} \to R$, and $\pi : R \oplus \mathfrak{g} \to R$ are maps which must satisfy certain properties in order for $R \oplus \mathfrak{g}$ to be a restricted Lie algebra. For example t must be a Lie 2-cocycle for $R \oplus \mathfrak{g}$ to be a Lie algebra (as in [4]); certain additional relations involving π must be satisfied for $R \oplus \mathfrak{g}$ to be restricted. These relations appear to be related to restricted Lie cocycles of $\mathfrak{g}$, though we need not deal with this aspect here (see [9]).

$(R, 0)$ is a copy of R which we identify with R. The image $\bar{\mathfrak{g}} = (0, \mathfrak{g})$ is twisted by t and $\pi : \bar{x}^{[p]} = \overline{x^{[p]}} + \pi(0, x)$ and $[\bar{x}, \bar{y}] = \overline{[x, y]} + t(x, y)$. Thus $(R \oplus \mathfrak{g})/R \cong \mathfrak{g}$.

Now $R * \mathfrak{g}$ is constructed as done in [4]. The difference here is that we take the restricted enveloping algebra of $R \oplus \mathfrak{g}$, take a factor ring, and end up with a ring $R * \mathfrak{g}$ with underlying k-space $R \otimes_k u(\mathfrak{g})$. The multiplication in $R * \mathfrak{g}$ is determined by the Lie bracket and the pth-power map in $R \oplus \mathfrak{g}$, and the multiplication in R.

$R * \mathfrak{g}$ is a free left and right R-module with basis consisting of restricted standard monomials in $\{\bar{x}_i\}$ where $x_1, x_2, \ldots$ forms a basis for $\mathfrak{g}$. Thus every $\alpha \in R * \mathfrak{g}$ is uniquely expressible as $\alpha = \sum a_\nu \bar{x}^\nu$, $a_\nu \in R$.

DEFINITION. Let A be an ideal of R. Define

$$(A:g) = \bigcap_{|\nu| \geqq 0} \delta^{-\nu}(A)$$

$$= \{a \in R \mid \delta^\nu(a) \in A \text{ for all } \nu\}.$$

Here ν ranges over dim $\mathfrak{g}$-tuples $(\nu_1, \nu_2, \ldots)$ with $0 \leqq \nu_i < p$. $(A:g)$ is the largest $\mathfrak{g}$-invariant ideal of R contained in A (here $\delta^{(0,0,\ldots)}(A) = A$).

R is said to be $\mathfrak{g}$-prime if $\mathfrak{g}$ acts on R and the product of $\mathfrak{g}$-invariant ideals is nonzero. $\mathfrak{g}$-prime ideals are $\mathfrak{g}$-invariant ideals with a $\mathfrak{g}$-prime factor ring.

The next lemma establishes a bijective, order-preserving correspondence between prime and $\mathfrak{g}$-prime ideals of R when $\mathfrak{g}$ is finite-dimensional.

LEMMA 1. *Let $\mathfrak{g}$ be finite-dimensional.*

(i) *Let Q be a $\mathfrak{g}$-prime ideal of R. Then there is a unique prime ideal $N(Q) \supset Q$ such that Q contains a power of $N(Q)$. Thus $N(Q)$ is the unique minimal prime containing Q and $(N(Q):\mathfrak{g}) = Q$.*

(ii) *Let P be a prime ideal of R. Then $(P:\mathfrak{g})$ is a $\mathfrak{g}$-prime ideal with $N((P:\mathfrak{g})) = P$.*

PROOF. We may assume that $Q = 0$ to prove (i). If δ is a derivation of R, and M an ideal of R then notice that $\delta(M^l) \subset M^{l-1}$ for all l. Thus for tuples ν, $\delta^\nu(M^l) \subseteq M^{l-|\nu|}$. Let $m = (p-1)$ dim $\mathfrak{g}$ and suppose $(M:\mathfrak{g}) = 0$. Then for all ν with $|\nu| \leqq m$ we have $\delta^\nu(M^{m+1}) \subset M$. Thus $M^{m+1} \subset (M:\mathfrak{g}) = 0$.

Let J be a nilpotent ideal of R. Since R is $\mathfrak{g}$-prime, we must have $(J:\mathfrak{g}) = 0$. Now applying the result of the preceding paragraph, we have $J^{m+1} = 0$.

What we have shown is that any nilpotent ideal is nilpotent of index at most $m+1$; thus if $N = N(0)$ denotes the sum of all nilpotent ideals of R, then $N^{m+1} = 0$. N is clearly the unique largest nilpotent ideal of R, and by the preceding paragraph N is the largest ideal with $(N:\mathfrak{g}) = 0$. Set $N = N(Q) = N(0)$.

To complete this proof of (i), let A_1 and A_2 be ideals of R strictly containing N. Then since N is maximal such that $(N:\mathfrak{g}) = 0$, $(A_i:\mathfrak{g}) \neq 0$ for $i = 1, 2$. By the

$\mathfrak{g}$-primeness of R, we see that $0 \neq (A_1 : \mathfrak{g})(A_2 : \mathfrak{g}) \subset AB$. Since the product of $\mathfrak{g}$-invariant ideals is $\mathfrak{g}$-invariant, we have $AB \not\subset N$. This shows that N is prime, and being nilpotent, N is surely a minimal prime.

Let A_1 and A_2 be $\mathfrak{g}$-invariant ideals of R with $A_1A_2 \subset (P : \mathfrak{g})$ where P is a prime ideal. Then $A_1A_2 \subset P$, so $A_i \subset P$ for some i. Hence $A_i = (A_i : \mathfrak{g}) \subset (P : \mathfrak{g})$, showing that $(P : \mathfrak{g})$ is $\mathfrak{g}$-prime. As in the proof of (i), P is nilpotent mod$(P : \mathfrak{g})$, so $N((P : \mathfrak{g})) = P$. This completes the proof of the lemma.

In particular, this lemma states that a $\mathfrak{g}$-prime ring has a unique largest nilpotent ideal, and this ideal is prime.

Next we mention some basic facts relating ideals of R and $R * \mathfrak{g}$. The following two easy lemmas are proved in the same way as for differential operator rings [4, §2].

Lemma 2. *Let A be a $\mathfrak{g}$-invariant ideal of R; then $A(R * \mathfrak{g}) = (R * \mathfrak{g})A$ is an ideal of $R * \mathfrak{g}$. Furthermore $A(R * \mathfrak{g}) \cap R = A$.*

This ideal shall be denoted $A * \mathfrak{g}$.

Lemma 3. *Let P be an ideal of $R * \mathfrak{g}$. Then $P \cap R$ is a $\mathfrak{g}$-invariant ideal of R. Furthermore if P is a prime ideal of $R * \mathfrak{g}$, then $P \cap R$ is a $\mathfrak{g}$-prime ideal of R.*

Conversely we have

Lemma 4. *Let A be a $\mathfrak{g}$-prime ideal of R; then there exists a prime ideal $P \subset R * \mathfrak{g}$ with $P \cap R = A$.*

Proof. By Lemma 2 there exists an ideal I with $I \cap R = A$. Applying Zorn's Lemma, there exists a maximal such ideal P, which is easily seen to be prime using Lemma 3.

Notice that $(R * \mathfrak{g})/(A * \mathfrak{g}) \cong (R/A) * \mathfrak{g}$, some twisted restricted differential operator ring over R/A. Thus when studying primes P with a fixed intersection with R we may pass to $(R/P \cap R) * \mathfrak{g}$ and assume that R is $\mathfrak{g}$-prime.

The quotient ring of a $\mathfrak{g}$-prime ring

In studying prime ideals of restricted differential operator rings $R * \mathfrak{g}$ we are led to the case where R is $\mathfrak{g}$-prime. It is then useful to construct a (Martindale) quotient ring of R.

Let $\mathcal{F} = \mathcal{F}(R)$ denote the set of $\mathfrak{g}$-invariant ideals of a $\mathfrak{g}$-prime ring R. As is done in [4], we use the two-sided quotient ring S of R with respect to $\mathcal{F}$. Set

$S = R_{\mathfrak{g}}$. This notation shall be fixed when R is $\mathfrak{g}$-prime, unless otherwise specified.

Briefly, $S = R_{\mathfrak{g}}$ can be described as equivalence classes of pairs of maps (f, g) where $f: A \to R$, $g: B \to R$ $(A, B \in \mathcal{F})$ are left and right module homomorphisms, respectively, satisfying $(af)b = a(gb)$ for $a \in A$ and $b \in B$. Two such pairs are defined to be equivalent if they agree on a common pair of domains. Alternatively, S can be described as the subring of the left ring of quotients consisting of elements s such that $sA \subset R$ for some $A \in \mathcal{F}$. This latter construction was used by Kharchenko in [11]. In fact the map induced by $(f, g) \to f$ embeds S into the left quotient ring. More information on this symmetric quotient ring may be found in [14].

The following lemma shows that S is large enough to contain all derivations which become inner in a one-sided quotient ring.

LEMMA 5. *Let D be a derivation of a $\mathfrak{g}$-prime ring R such that $D(A) \subset A$ for all $A \in \mathcal{F}$. If there exists d, an element of the left ring of quotients of R, such that*

$$D(r) = [d, r]$$

for all $r \in R$, then $d \in S$, the two-sided quotient ring of R.

PROOF. Since d is an element of the quotient ring, there exists $A \in \mathcal{F}$ with $Ad \subset R$. Observe that

$$D(a) = da - ad$$

for $a \in A$. Since $D(A) \subset A$, it follows that $dA \subset R$. Hence $d \in S$ using the second description of the two-sided quotient ring above.

Basic properties of subgrings of the quotient ring are listed in the next lemma. See [4, §1] for proofs.

LEMMA 6. *Let R be a $\mathfrak{g}$-prime ring.*

(i) *R is embedded in S via left and right multiplication on R.*

(ii) *Let $s_1, \ldots, s_n$ be elements of S. Then there exists $A \in \mathcal{F}$ with $s_iA \subset R$ and $As_i \subset R$ for all i. If either $As_i = 0$ or $s_iA = 0$, then $s_i = 0$.*

(iii) *Let $s \in S$ be an element of the two-sided quotient ring represented by $f: {}_RA \to R$ and $g: A_R \to R$. Then $af = as$ and $ga = sa$ for all $a \in A$.*

(iv) *The derivations δ_x, $x \in \mathfrak{g}$ extend uniquely to derivtions of S (denoted by the same symbol).*

DEFINITION. Let R be a $\mathfrak{g}$-prime ring.

(i) The *extended center* of R is defined to be the center of S.

(ii) The $\mathfrak{g}$-*extended center* of R is the subring of the extended center of R consisting of elements vanishing under the action of $\mathfrak{g}$.
(iii) R is said to be $\mathfrak{g}$-*centrally closed* if R contains its $\mathfrak{g}$-extended center.

Let C denote the extended center of R and let F denote $C^{\mathfrak{g}}$, the subring of constants relative to $\mathfrak{g}$ (= the $\mathfrak{g}$-extended center of R).

LEMMA 7. *F is a field.*

PROOF. Let q be a nonzero element of F and let $A \in \mathscr{F}$ with $0 \neq Aq \subset R$. Let $f: A \to R$ denote multiplication by q. Since q is central and $\delta_x(q) = 0$, $x \in \mathfrak{g}$, $qA = Aq \in \mathscr{F}$.

Note that $K = \ker f$ is a $\mathfrak{g}$-invariant ideal of R. Since $Kq = 0$ and $q \neq 0$, we have $K = 0$. Thus f has an inverse $f^{-1}: Aq \to A$. It follows that q is invertible.

We remark that C need not be a field. However, if $\mathfrak{g}$ is finite dimensional, then C is a local ring finite dimensional over F. In fact, if $\mathfrak{g}$ acts faithfully on C, then $C * \mathfrak{g}$ is simple, End $C_{C*\mathfrak{g}} = F$, $C * \mathfrak{g}$ is Morita equivalent to F, and thus C is finitely generated as an F-module [2]. C is a $\mathfrak{g}$-prime ring whose maximal ideal is nilpotent by Lemma 1.

The following proposition is a slight modification of [4, Theorem 1.4].

PROPOSITION 8. *Let R be a $\mathfrak{g}$-prime ring. Then S is a $\mathfrak{g}$-prime ring which is $\mathfrak{g}$-centrally closed.*

PROOF. It follows from Lemma 6(ii) that every nonzero $\mathfrak{g}$-invariant ideal of S contains a nonzero $\mathfrak{g}$-invariant ideal of R. Thus, it follows that S is $\mathfrak{g}$-prime.

We must show that the $\mathfrak{g}$-extended center of S is contained in S, so let z be a nonzero element of the $\mathfrak{g}$-extended center. The element z has the properties that $\delta_x(z) = 0$ ($x \in \mathfrak{g}$) and $0 \neq zI = Iz \subset S$ for some nonzero $\mathfrak{g}$-invariant ideal $I \subset S$.

Since z is central Iz is an ideal of S, and because the action of $\mathfrak{g}$ is trivial on z and I is $\mathfrak{g}$-invariant,

$$\delta_x(sz) = \delta_x(s)z, \qquad x \in \mathfrak{g};$$

thus Iz is $\mathfrak{g}$-invariant.

Using Lemma 6(ii), we see that $Iz \cap R$ is a nonzero, $\mathfrak{g}$-invariant ideal of R; therefore

$$J = \{a \in I \mid az \in R\} \neq 0.$$

Note that J is a nonzero $R-R$ bimodule contained in I. In addition $\delta_x(z)=0$ $(x\in\mathfrak{g})$ yields that J is $\mathfrak{g}$-invariant.

Let $A=J\cap R$. Then A is a nonzero $\mathfrak{g}$-invariant ideal of R, that is, $A\in\mathscr{F}$. Define a map $g: A\to R$ by

$$ag=az, \qquad a\in A.$$

Since z is central in S, g is clearly an $R-R$ bimodule map, so g represents an element c of C, the center of S. Now $ag=ac=az$ for all $a\in A$.

Observe that $A(c-z)=0$. We finish the proof by showing that $c=z$; for then $\delta_x(c)=\delta_x(z)=0$, so $z=c\in F$. Let B be a nonzero $\mathfrak{g}$-invariant ideal of S with $B(c-z)\subset S$. Let $q\in B(c-z)$ and let I be a $\mathfrak{g}$-invariant ideal of R with $Iq\subset R$. Then, since $c-z$ centralizes A, we have $IqA=0$. Since $A\in\mathscr{F}$ and R *is* $\mathfrak{g}$-invariant, $Iq=0$, so $q=0$. Thus $B(c-z)=0$ so $c-z=0$. This completes the proof.

Restricted differential operator rings over quotient rings

Let R be $\mathfrak{g}$-prime and let $S=R_{\mathfrak{g}}$ be its quotient ring. The action of $\mathfrak{g}$ on R extends uniquely to an action $\delta:\mathfrak{g}\to \mathrm{Der}_k(S)$ by Lemma 6(iii). With this extension we form $S*\mathfrak{g}$ by extending coefficients. One must check that the twisting given by t and π makes $S*\mathfrak{g}$ an associative ring. One way to do this is to write out the relations on δ, π and t which are equivalent to the associativity of $R*\mathfrak{g}$, and then check that $S*\mathfrak{g}$ is associative with the same twistings.

Alternatively one may start out with the twisted semidirect product $S\oplus\mathfrak{g}$, and show that $S\oplus\mathfrak{g}$ is a restricted Lie algebra; then $S*\mathfrak{g}$ may be constructed as done for $R*\mathfrak{g}$.

A less tedious way of checking the associativity of $S*\mathfrak{g}$ is as follows. Let $\mathscr{F}^*=\{A*\mathfrak{g}\mid A\in\mathscr{F}\}$ where $\mathscr{F}$ denotes the set of nonzero $\mathfrak{g}$-invariant ideals of R. $\mathscr{F}^*$ consists of ideals having zero left and right annihilators, so we may construct the left quotient ring T of $R*\mathfrak{g}$ with respect to $\mathscr{F}^*$. As usual $R*\mathfrak{g}$ embeds in T by right multiplications, and S embeds by the multiplications $\Sigma\,\bar{x}^{\nu}a_{\nu}\to\Sigma\,\bar{x}^{\nu}(a_{\nu}s)$, where $\Sigma\,\bar{x}^{\nu}a_{\nu}\in A*\mathfrak{g}$, $s\in S$ and $As\subset R$. Then one checks that the subring of T generated by S and $R*\mathfrak{g}$ is precisely $S*\mathfrak{g}$. Since T is associative, so too is $S*\mathfrak{g}$.

Definition 9. Let R be a $\mathfrak{g}$-prime ring, and let $R*\mathfrak{g}\subset S*\mathfrak{g}$ be the extension to S, the two-sided quotient ring of R. Let I and J denote ideals of $R*\mathfrak{g}$ and $S*\mathfrak{g}$, respectively.

(i) Define

$$I^U = \{\alpha \in S * \mathfrak{g} \mid A\alpha B \subset I \text{ for some } A, B \in \mathcal{F}\}$$

and

(ii)

$$J^D = J \cap (R * \mathfrak{g}).$$

It follows easily that I^U and J^D are ideals of $S * \mathfrak{g}$ and $R * \mathfrak{g}$, respectively (see [4, Lemma 2.9]).

Relating ideals in $R * \mathfrak{g}$ and ideals in $S * \mathfrak{g}$, we have the following partial correspondence, which shall be improved in Theorem 20 when $\mathfrak{g}$ is finite dimensional abelian.

Lemma 10. *Let $R * \mathfrak{g}$ be a restricted differential operator ring with R a $\mathfrak{g}$-prime ring. Consider the extension $R * \mathfrak{g} \subset S * \mathfrak{g}$. If P is a prime ideal of $R * \mathfrak{g}$ with $P \cap R = 0$, then P^U is a prime ideal of $S * \mathfrak{g}$ with $P^{UD} = P$ and $P^U \cap S = 0$.*

Proof. Versions of this lemma are proved for crossed products and differential operator rings in [4, Lemma 2.9, Lemma 3.3]. This lemma is proved in the same way.

Consider the restricted differential operator ring $S * \mathfrak{g}$. Extending the field of scalars, set $\mathfrak{g}^F = \mathfrak{g} \otimes F$. Then the action $\delta : \mathfrak{g} \to \mathrm{Der}\, S$ extends in the natural fashion to $\delta : \mathfrak{g}^F \to \mathrm{Der}_F(S)$. It is also easy to see that $C^{\mathfrak{g}^F} = C^{\mathfrak{g}} = F$. Note that the elements $\bar{x}_1, \bar{x}_2, \ldots \in S * \mathfrak{g}$ are S-linear independent, so their F-linear span is clearly $\bar{\mathfrak{g}}^F$. Hence $S * \mathfrak{g} = S * \mathfrak{g}^F$, a restricted differential operator ring of $\mathfrak{g}^F$ over S.

Outer actions

$R * \mathfrak{g}$ is said to have the *ideal intersection property* if every nonzero ideal of $R * \mathfrak{g}$ has nonzero intersection with R. The next result is similar to [2, Theorem 1.2] in which R is assumed to be prime, but $\mathfrak{g}$ need not be abelian.

Theorem 11. *Let $R * \mathfrak{g}$ be a restricted differential operator ring with R a $\mathfrak{g}$-prime ring and $\mathfrak{g}$ abelian. If no nonzero F-linear combination of elements of $\mathfrak{g}$ is inner in S, then $R * \mathfrak{g}$ has the ideal intersection property.*

Proof. Let I be a nonzero ideal of $R * \mathfrak{g}$ and suppose that $I \cap R = 0$. Fix a basis $\{x_i\}$ for $\mathfrak{g}$. Let m be minimial among the total degrees of nonzero elements of I. Our assumption on I implies that $m > 0$. Let V denote the set of $\dim \mathfrak{g}$-tuples ν with $|\nu| = m$. Further, let W denote a subset of V of minimal size subject to the condition that there is a nonzero element $\alpha \in I$ with $|\alpha| = m$, where $\mathrm{Supp}_m(\alpha) = \bar{x}^W = \{\bar{x}^\nu \mid \nu \in W\}$.

Define for dim $\mathfrak{g}$-tuples ν,

$$A_\nu = \{r \in R \mid \text{there exists } \alpha = \Sigma\, a_\xi \bar{x}^\xi \in I \text{ with}$$
$$a_\nu = r, \mathrm{Supp}_m(\alpha) \subset \bar{x}^W \text{ and } |\alpha| \leqq m\}.$$

Observe that for $y \in \mathfrak{g}$, α as in the definition of A_ω, and $\omega \in W$,

$$[\bar{y}, \alpha] = \sum_{\mu \in W} \delta_y(a_\mu)\bar{x}^\mu + \alpha_- \in I, \qquad |\alpha_-| < m,$$

using the fact that $\mathfrak{g}$ is abelian. It follows that $\delta_y(a) \in A_\omega$ for $a \in A_\omega$ and thus $A_\omega \in \mathcal{F}$.

Fix $\omega \in W$ and let $A = A_\omega$. We may assume that our basis was chosen so that $\omega = (\omega_1, \ldots, \omega_l)$ with $\omega_1 > 0$. Define maps $f_\nu : A \to A_\nu$ as follows. Let $a \in A$ and let α be as in the definition of $A = A_\omega$. It follows from the minimality of W that this α is unique. Write $\alpha = \Sigma\, a_\nu \bar{x}^\nu$ and define $a f_\nu = a_\nu$. The f_ζ are easily seen to be left R-module maps. Furthermore, if $\zeta \in W$, f_ζ is actually an $R-R$ bimodule map. Note that f_ω is the identity map.

Let $\zeta \in W$ and let c_ζ be the element of the extended center of R represented by f_ζ. Note that $c_\omega = 1$. Let us drop subscripts and set $c = c_\zeta$, $f = f_\zeta$. We claim that $c \in F$, the $\mathfrak{g}$-extended center of R. To see this, let $y \in \mathfrak{g}$ and note that, by the formula for $[\bar{y}, \alpha]$ above and the definition of f, we have

$$\delta_y(a) f = \delta_y(a_\zeta) = \delta_y(a f).$$

By Lemma 6(iii),

$$\delta(a)c = \delta(ac).$$

Thus we conclude that $a\delta(c) = 0$, so $A\delta(c) = 0$. Therefore by Lemma 6(ii), $\delta(c) = 0$, showing that $c \in F$.

Set $\omega' = (\omega_1 - 1, \ldots, \omega_n)$. (Recall that $\omega_1 > 0$.) Using $a \in A$ and α as above, compute for $r \in R$

$$\alpha r = \sum_{\mu \in W} a_\mu \bar{x}^\mu r + a_{\omega'} \bar{x}^{\psi'} + \cdots$$
$$= \sum_{\mu \in W} ((a_\mu r)\bar{x}^\mu + a_\mu[\bar{x}^\mu, r]) + a_{\omega'} r \bar{x}^{\omega'} + \cdots$$

where we omit terms of degree less than m except for the $\bar{x}^{\omega'}$ term. Given $i > 0$, let $\mu = \mu(i)$ be the element of V with $\mu_i = \omega_i' + 1$. In particular $\mu(1) = \omega$.

Observe that the coefficient of $\bar{x}^{\omega'}$ in αr depends on these i's and in fact, this coefficient is

$$a_{\omega'}r + \sum_i a_\mu \mu_i \delta_i(r).$$

Now by the definition of $f_{\omega'}$ we have

$$(ar)f_{\omega'} = (af_{\omega'})r + \sum_i (af_\mu)\mu_i \cdot \delta_i(r).$$

Furthermore letting s denote the element of the left quotient ring of R represented by $f_{\omega'}$, we see that

$$ars = a\left(sr + \sum_i c_\mu \mu_i \cdot \delta_i(r)\right).$$

Thus

$$A\left(rs - sr - \sum_i (c_\mu \mu_i)\cdot \delta_i(r)\right) = 0,$$

so we see that $[s,\ \ ] = \Sigma_i \mu_i c_\mu \delta_i$ as a derivation of R (and hence of S). Hence $\Sigma_i c_\mu \mu_i \delta_i$ is inner in the left quotient ring of R. But now Lemma 6 applies to show that $s \in S$.

We saw above that $c_\mu \in F$ since $\mu \in W$, so $\Sigma_i (c_\mu \mu_i)x_i$ is an F-linear combination of $x_i \in \mathfrak{g}$ which is inner in S. This linear combination is nonzero because when $i = 1$, we have $\mu = \omega$ $c_\mu = 1$ and $0 < \mu_1 < p$, so the proof is complete.

This theorem has a few immediate consequences.

Corollary 12. *Let $R * \mathfrak{g}$ be as in the theorem. Then $R * \mathfrak{g}$ is a prime ring.*

Corollary 13. *Let $R * \mathfrak{g}$ be as in the theorem and consider the extension to $S * \mathfrak{g}$. Then $S * \mathfrak{g}$ has the ideal intersection property.*

Proof. Observe that every nonzero ideal of $S * \mathfrak{g}$ intersects $R * \mathfrak{g}$ in a nonzero ideal (Lemma 6(iii)), which, by the ideal intersection property, has nonzero intersection with R and hence with S.

Let $\mathfrak{i} = \mathfrak{i}(\mathfrak{g}^F)$ denote the restricted Lie ideal of $\mathfrak{g}^F$ consisting of the F-linear combinations of elements of $\mathfrak{g}$ whose action on S is inner in S. In particular if $\mathfrak{i} = 0$, then the hypotheses of the theorem are satisfied. Hence we have

COROLLARY 14. *Let $R*\mathfrak{g} \subset S*\mathfrak{g}$ be as above; if $\mathfrak{i}=0$, then both $R*\mathfrak{g}$ and $S*\mathfrak{g}$ have the ideal intersection property.*

Inner actions

Let R be $\mathfrak{g}$-prime. Again let $\mathfrak{i}$ denote the ideal of $\mathfrak{g}^F$ consisting of elements of $\mathfrak{g}^F$ which become inner in S. Accordingly let d_x be an element of S such that $[d_x, \ ]=\delta_x$ for each $x \in \mathfrak{i}$.

For the following several lemmas, through Theorem 17, we assume that $\mathfrak{i}=\mathfrak{g}^F$, that is, the action of $\mathfrak{g}^F$ on S is inner in S. It is immediate that $\mathfrak{i}=\mathfrak{g}^F$ is equivalent to the action of $\mathfrak{g}$ on S being inner.

Note that $F=C$, the extended center of R since the action of $\mathfrak{g}$ and $\mathfrak{g}^F$ is trivial on C, the center of S. Also S is a prime ring because all ideals of S are $\mathfrak{g}$-invariant, and F is a field by Lemma 7.

Let $R*\mathfrak{g}$ be a restricted differential operator ring and let $S*\mathfrak{g}$ be the extension. Recall that $S*\mathfrak{g}=S*\mathfrak{g}^F$.

LEMMA 15. *Let E be the centralizer of S in $S*\mathfrak{g}$. Then $S*\mathfrak{g}=S\otimes_F E$, and $E=F*\mathfrak{g}$ is a twisted restricted enveloping algebra over the field F where $\mathfrak{g}$ acts trivially on F.*

PROOF. As above let $\tilde{x}=\bar{x}-d_x$ for $x \in \mathfrak{g}$. Then $\tilde{x}\in E$, and (restricted) standard monomials in the $\{\tilde{x}_i\}$ form a free basis for $S*\mathfrak{g}$ over S, where $\{x_i\}$ is a basis for $\mathfrak{g}$. Hence $S*\mathfrak{g}=SE$.

Next suppose $\alpha \in E$. Then $\alpha=\Sigma\, a_\nu \tilde{x}^\nu$ with $a_\nu \in S$. Since the $\tilde{x}^\nu$ centralize S, as does α, we deduce that $a_\nu \in S\cap E=F$. Thus the $\tilde{x}_i^\nu$ form a F basis for E.

Next observe that for $x, y \in \mathfrak{g}$,

$$[\tilde{x},\tilde{y}]=[x,y]\tilde{\ }+\tilde{t}(x,y),$$

where $\tilde{t}(x,y)$ is some element of S and hence of F. Also

$$\tilde{x}^p=\tilde{x}^{[p]}+\tilde{\pi}(x),$$

where $\tilde{\pi}(x)$ is an element of F. Since E is an associative F-algebra with an F-basis consisting of standard monomials in the $\tilde{x}_i$, it follows that E is a restricted enveloping algebra over F, twisted by $\tilde{t}$ and $\tilde{\pi}$.

Finally let $S\otimes_F E \to SE=S*\mathfrak{g}$ be the usual map. Note that each $\alpha \in S\otimes E$ may be written uniquely as $\alpha=\Sigma\, a_\nu \otimes \tilde{x}^\nu$ where $a_\nu \in S$; thus α is mapped to $\Sigma\, a_\nu\tilde{x}^\nu \in S*\mathfrak{g}$, which is nonzero unless $\alpha=0$. Thus $S\otimes E=S*\mathfrak{g}$.

We now shift our attention to a more general setting. Let S be a centrally closed prime ring with center F and let $S \otimes_F E$ be a ring with a subring E centralizing S. The next theorem, which is probably well known, describes a bijective correspondence between prime ideals of $S \otimes E$ having zero intersection with S and the prime ideals of E. In view of the preceding lemma, this result yields information on primes in $S * \mathfrak{g}$ when the action of $\mathfrak{g}$ in inner in S. Let $\mathscr{F}$ denote the set of nonzero ideals of S, and let $\otimes$ denote $\otimes_F$.

Since $S \otimes E$ is a centralizing extension of S we could make use of results in [15]; however, our treatment is more direct relative to the study of $S \otimes E$.

Lemma 16. *Let S be a centrally closed prime ring which is a subring of $S \otimes E$ where F is the center of S. Then every nonzero ideal of $S \otimes E$ contains an ideal $A \otimes B$ where A and B are nonzero ideals of S and E respectively.*

Proof. Let I be a nonzero ideal of $S \otimes E$. Let $\alpha = \Sigma_1^n a_i \otimes e_i$ be an element of I of minimal length n among such expressions of nonzero elements of I. Fix the $e_1, \dots, e_n \in E$ which occur in α.

Define ideals $A = B_1, B_2, \dots, B_n$ as follows. Let $B_j = \{s \in S \mid \text{there exists } \Sigma_1^n b_i \otimes e_i \in I \text{ with } b_j = s\}$. The B_j are nonzero ideals of R, and for each $a \in A$ there is a unique element $\beta = \Sigma\, a_i \otimes e_i \in I$ with $a_1 = a$. Uniqueness follows from the minimality of n. Thus we can define maps $f_j : A \to B_j$ by $a f_j = a_j$ for all $a \in A$. Since E commutes with S, the f_j are $S - S$ bimodule homomorphisms. Let c_j be the element of the extended center of S represented by f_j. Then $c_j \in F$ because S is centrally closed, and also $c_j \neq 0$.

Define

$$\beta = \sum_i^n c_i \otimes e_i \in E.$$

If $a \in A$, there exists $\Sigma\, a_i \otimes e_i$ as in the definition of A. By construction, $a\beta = \Sigma\, ac_i \otimes e_i = \Sigma\, a f_i \otimes e_i = \Sigma\, a_i \otimes e_i$. Hence $A\beta \subset I$.

Since E commutes with S, we see that

$$(S \otimes E)A\beta(S \otimes E) = A \otimes E\beta E$$

is a nonzero ideal of $S \otimes E$ containing I. Setting $B = E\beta E$, we have the desired conclusion.

Theorem 17. *Let S, E and $S \otimes E$ be as in the lemma above. Let P be a prime ideal $S \otimes E$ with $P \cap S = 0$, and let L be a prime ideal of E. Then*

(i) *$P \cap E$ is a prime ideal of E with $S \otimes (P \cap E) = P$.*

(ii) *$S \otimes L$ is a prime ideal of $S \otimes E$ with $(S \otimes L) \cap E = L$ and $(S \otimes L) \cap S = 0$.*

PROOF. (i) Let us first show that either $P = 0$ or $P \cap E \neq 0$. Suppose $P \neq 0$. By the previous lemma there exist nonzero ideals $A \subset S$ and $B \subset E$ with $A \otimes B \subset P$. Since E commutes with S, we have $(A \otimes E)(S \otimes B) = A \otimes B \subset P$. Since P is prime, and $A \not\subset P$ (because $P \cap S = 0$), it follows that $0 \neq B \subset P \cap E$.

Let $\bar{E} = E/(P \cap E)$ and note that $S \otimes \bar{E} = (S \otimes E)/(S \otimes (P \cap E))$. Consider the map $S \otimes \bar{E} \to (S \otimes E)/P$. The kernel of this map is a prime ideal which has zero intersection with both S and $\bar{E}$. Applying the result of the previous paragraph to $S \otimes \bar{E}$, we see that the kernel is zero. Hence $P = S \otimes (P \cap E)$.

To show that $P \cap E$ is a prime ideal of E, let B_1 and B_2 be ideals of E with $B_1B_2 \subset P \cap E$. Since E centralizes S, we have $(S \otimes B_1)(S \otimes B_2) = S \otimes B_1B_2 \subset S \otimes (P \cap E) = P$. Thus the primeness of P forces $S \otimes B_i \subset P$ for some i. Intersecting with E yields $B_i \subset P \cap E$, so $P \cap E$ is prime.

(ii) We show that $S \otimes L$ is prime, the remaining assertions being trivial. Set $\bar{E} = E/L$. Then $S \otimes \bar{E} \cong S \otimes E/S \otimes L$. Let I_1 and I_2 be nonzero ideals of $S \otimes \bar{E}$. Let $A_i \otimes B_i$ be as in the lemma, with $A_i \otimes B_i \subset I_i$, and A_i, B_i nonzero, $i = 1, 2$. Observe that $I_1I_2 \supset (A_1 \otimes B_1)(A_2 \otimes B_2) = A_1A_2 \otimes B_1B_2$. Since both S and $\bar{E}$ are prime, we obtain $I_1I_2 \neq 0$. Thus $S \otimes \bar{E}$ is a prime ring, so $S \otimes L$ is a prime ideal of $S \otimes E$.

Incomparability

It is convenient to note the following lemma.

LEMMA 18. *Let $R * \mathfrak{g}$ be a restricted differential operator ring and suppose $\mathfrak{i}$ is a restricted ideal of $\mathfrak{g}$ of finite codimension. Let $P_1 \subset P_2$ be prime ideals of $R * \mathfrak{g}$. Then there exist prime ideals $Q_1 \subset Q_2$ of $R * \mathfrak{i}$ such that*

(i) *Q_i is the unique minimal prime ideal of $R * \mathfrak{i}$ containing $P_i \cap (R * \mathfrak{i})$, $i = 1, 2$.*

(ii) *If $Q_1 = Q_2$, then $P_1 \cap (R * \mathfrak{i}) = P_2 \cap (R * \mathfrak{i})$.*

(iii) *If $P_1 \cap R = P_2 \cap R$, then $Q_1 \cap R = Q_2 \cap R$.*

PROOF. Observe that $R * \mathfrak{g} = (R * \mathfrak{i}) * \tilde{\mathfrak{g}}$, some restricted differential operator ring of $\tilde{\mathfrak{g}} = \mathfrak{g}/\mathfrak{i}$, a restricted Lie algebra of finite dimension.

Set $A_i = P_i \cap (R * \mathfrak{i})$ and note that the A_i are $\mathfrak{g}$-prime ideals of $R * \mathfrak{i}$. From Lemma 1 we see that there exist prime ideals $Q_i \subset R * \mathfrak{i}$ satisfying (i). In fact, by that lemma, A_i contains a power of Q_i and $A_i = (Q_i : \tilde{\mathfrak{g}})$. It now follows that $Q_1 \subset Q_2$ and that (ii) is satisfied.

To establish (iii), suppose that $P_1 \cap R = P_2 \cap R$. We have $A_i \cap R = P_i \cap R$, and $Q_i \subset A_i$; therefore, for some m,

$$(Q_1 \cap R)^m \subset P_2 \cap R = P_1 \cap R \subset Q_1 \cap R.$$

Since the Q_i are $\mathfrak{i}$-invariant and $Q_1 \cap R$ is an $\mathfrak{i}$-prime ideal of R, we have $Q_2 \cap R \subset Q_1 \cap R$.

THEOREM 19. (Incomparability). *Let $R * \mathfrak{g}$ be a restricted differential operator ring with $\mathfrak{g}$ finite dimensional abelian. If $P_1 \subsetneqq P_2$ are prime ideals of $R * \mathfrak{g}$, then $P_1 \cap R \neq P_2 \cap R$.*

PROOF. By passing to $R/P_1 \cap R$ we may assume that R is a $\mathfrak{g}$-prime ring and $P_1 \cap R = 0$. We must show that $P_2 \cap R \neq 0$.

We apply Lemma 10 and its notation to obtain ideals P_1^U and P_2^U of $S * \mathfrak{g}$. If $P_2 \cap R = 0$ we have $P_1^U \subsetneqq P_2^U$, $P_i^U \cap S = 0$, and P_i^U is a prime ideal of $S * \mathfrak{g}$ ($i = 1, 2$). Thus it suffices to show that $P_2^U \cap S \neq 0$.

In view of the observation of the last paragraph, we may start with fresh notation. Let $P_1 \subsetneqq P_2$ be prime ideals of $S * \mathfrak{g}$ with $P_1 \cap S = 0$. We shall show that $P_2 \cap S \neq 0$.

As previously mentioned we have $S * \mathfrak{g} = S * \mathfrak{g}^F$, $\mathfrak{g}^F = \mathfrak{g} \otimes F$. Also $\dim_F \mathfrak{g}^F = \dim \mathfrak{g}$, and we let $\mathfrak{i}$ denote the restricted ideal of $\mathfrak{g}^F$ consisting of elements that are inner in S.

First consider the case $\mathfrak{i} = \mathfrak{g}^F$. Then $S * \mathfrak{g} = S \otimes_F E$ where E is a finite dimensional algebra over the field F (Lemma 15). Using Theorem 17, $P_1 \cap E$ is a prime ideal of E, which is maximal since E is finite dimensional. The ideal correspondence in that theorem immediately yields $P_2 \cap S \neq 0$.

Next suppose $\mathfrak{i} = 0$. Then the hypothesis of Corollary 14 is satisfied, so again, $P_2 \cap S \neq 0$.

In the remaining case $0 \neq \mathfrak{i} \neq \mathfrak{g}^F$, we have $\dim_F \mathfrak{i} < \dim_F \mathfrak{g}^F$. Notice that we may write

$$S * \mathfrak{g} = S * \mathfrak{g}^F = (S * \mathfrak{i}) * \tilde{\mathfrak{g}},$$

some restricted differential operator ring of $\tilde{\mathfrak{g}} = \mathfrak{g}^F/\mathfrak{i}$ over the ring $S * \mathfrak{i} = T$.

Since $\dim_F \tilde{\mathfrak{g}} < \dim \mathfrak{g}$, induction yields

$$P_1 \cap T \neq P_2 \cap T.$$

Note that the $P_i \cap T$ are $\tilde{\mathfrak{g}}$-prime ideals of T. Using Lemma 18 there exist prime ideals $Q_i \subset T$ with $Q_i \supset P_i \cap T$ and $Q_1 \subsetneqq Q_2$. Since $\dim_F \mathfrak{i} < \dim \mathfrak{g}$, induction also yields

$$Q_1 \cap S \neq Q_2 \cap S;$$

thus by Lemma 18(iii) $P_2 \cap S \neq P_1 \cap S = 0$.

This completes the proof of Theorem 19.

Minimal primes and going down

Recall the maps ^{U} and ^{D} considered in Lemma 10. The next result shows that these maps are inverses on primes having trivial intersection with the coefficient ring, provided $\mathfrak{g}$ is finite dimensional abelian.

THEOREM 20. *Let $R * \mathfrak{g}$ be a restricted differential operator ring with $\mathfrak{g}$ finite dimensional and abelian, and R a $\mathfrak{g}$-prime ring. The maps ^{U} and ^{D} in Definition 9 set up a one-to-one correspondence between the prime ideals of $R * \mathfrak{g}$ having zero intersection with R and the primes of $S * \mathfrak{g}$ having zero intersection with S. Precisely*:

(i) *If P is prime ideal of $R * \mathfrak{g}$ with $P \cap R = 0$, then P^U is a prime ideal of $S * \mathfrak{g}$ with $P^U \cap S = 0$, and $P^{UD} = P$.*

(ii) *If I is a prime ideal of $S * \mathfrak{g}$ with $I \cap S = 0$, then I^D is a prime ideal of $R * \mathfrak{g}$ with $I^D \cap R = 0$ and $I^{DU} = I$.*

PROOF. Let I be as in the statement of the theorem. In view of Lemma 10 it suffices to establish the conclusions concerning I.

By an application of Zorn's Lemma, there exists an ideal $Q \subset R * \mathfrak{g}$ maximal subject to $I^D \subset Q$ and $Q \cap R = 0$. It follows from the $\mathfrak{g}$-primeness of R that Q is a prime ideal. By Lemma 10, $Q^{UD} = Q$ and Q^U is a prime ideal of $S * \mathfrak{g}$ with $Q^U \cap S = 0$. The theorem will follow once we show that $Q^U = I$, for then $Q = Q^{UD} = I^D$ and, consequently, $I^{DU} = Q^U = I$.

Let $\alpha \in I$. There is an ideal $A \in \mathcal{F}$ such that $A\alpha \subset I \cap (R * \mathfrak{g}) = I^D$. It follows immediately from the definition of ^{U} that $\alpha \in I^{DU}$. Thus $I \subset I^{DU} \subset Q^U$. Since $I \subset Q^U$ are prime ideals, both having trivial intersection with S, Theorem 19 yields $I = Q^U$.

Induction can now be used to describe minimal primes in $R * \mathfrak{g}$ when R is $\mathfrak{g}$-prime and $\mathfrak{g}$ is finite dimensional abelian. "Going Down" is an immediate corollary.

THEOREM 21. *Let $R * \mathfrak{g}$ be as in the preceding theorem and let $\mathcal{P}$ denote the set of minimal primes of $R * \mathfrak{g}$, and set $n = \dim \mathfrak{g}$. Then*

(i) *$P \in \mathscr{P}$ if and only if $P \cap R = 0$,*

(ii) $|\mathscr{F}| \leqq \mathscr{P}^n$,

(iii) $(\cap \mathscr{P})^{p^n} = 0$, *and* $\cap \mathscr{P}$ *is the unique largest nilpotent ideal of* $R * \mathfrak{g}$.

Proof. Suppose first that $R * \mathfrak{g}$ has a finite set of primes $\mathscr{P}$ with nilpotent interesection and $P \cap R = 0$ for all $P \in \mathscr{P}$. Then, by Incomparability, each $P \in \mathscr{P}$ is surely a minimal prime. Also, the nilpotence of $\cap \mathscr{P}$ guarantees that the minimal primes of $R * \mathfrak{g}$ form a subset of $\mathscr{P}$. Thus $\mathscr{P}$ is precisely the set of minimal primes of $R * \mathfrak{g}$. Furthermore, being the intersection of a finite set of primes, $\cap \mathscr{P}$ contains every nilpotent ideal. Thus to prove the theorem it suffices to establish the existence of such a set $\mathscr{P}$ satisfying the nilpotence and cardinality statements (ii) and (iii).

Let us reduce to consideration of the extension $S * \mathfrak{g}$, i.e., we claim that if $S * \mathfrak{g}$ satisfies the conclusions of the theorem, then so does $R * \mathfrak{g}$. For suppose $S * \mathfrak{g}$ satisfies the conclusions of the theorem with minimal primes $\mathscr{P}$; then by Theorem 20, we see that $\mathscr{P}^D = \{P^D \mid P \in \mathscr{P}\}$ is a set of primes of $R * \mathfrak{g}$ having zero intersection with R, having size at most p^n, and which satisfies $(\cap \mathscr{P}^D)^{p^n} \subset (\cap \mathscr{P})^{p^n} = 0$. As in the first paragraph we conclude that $\mathscr{P}^D$ is precisely the set of minimal primes of $R * \mathfrak{g}$ and evidently satisfies the conclusions of the theorem.

We proceed by induction on $n = \dim \mathfrak{g} = \dim_F \mathfrak{g}^F$. We shall write $\mathfrak{g}$ for $\mathfrak{g}^F$ for the remainder of the proof. Let $\mathfrak{i}$ denote the ideal of $\mathfrak{g}$ consisting of elements which are inner in S.

If $\mathfrak{i} = 0$, Theorem 11 and Corollary 12 apply to show that $S * \mathfrak{g}$ is prime, so the conclusions of the theorem are trivially satisfied.

On the other hand if $\mathfrak{i} = \mathfrak{g}$, we may apply Theorem 17 and its notation. Here $S * \mathfrak{g} = S \otimes_F E$ where E is a p^n-dimensional algebra over F. Let $\mathscr{L}$ be the set of minimal primes of E. Then $|\mathscr{L}| \leqq p^n$, $(\cap \mathscr{L})^{p^n} = 0$, and $\mathscr{L}$ is the radical of E. By Theorem 17 we obtain the set $\mathscr{P} = \{S \otimes L \mid L \in \mathscr{L}\}$ of primes of $S \otimes E$ of size at most p^n, having zero intersection with S. And since E centralizes S, $(\cap \mathscr{P})^{p^n} = 0$. Hence we are done in this case.

In the remaining case we have $0 \neq h \neq \mathfrak{g}$. Let $\tilde{\mathfrak{g}} = \mathfrak{g}/i$, $i = \dim \mathfrak{i}$, and $j = \dim \tilde{\mathfrak{g}} = n - i$. Also let T denote $S * h$ so that $S * \mathfrak{g} = T * \tilde{\mathfrak{g}}$.

Let $N_\mathfrak{g}$ be the unique minimal prime of S as described by Lemma 1. Let $N = (N_\mathfrak{g} : \mathfrak{i})$ and note that N is an $\mathfrak{i}$-prime ideal of S. Consider $T/(N * \mathfrak{i}) = (S/N) * \mathfrak{i}$. Since $\dim \mathfrak{i} = i < n$, induction yields a set of primes $\mathscr{Q}'$ of T, minimal over $N * \mathfrak{i}$, such that $Q' \in \mathscr{Q}'$ if and only if

$$Q' \cap S = N, \quad |\mathscr{Q}'| \leqq p^i \quad \text{and} \quad (\cap \mathscr{Q}')^{p^i} \subset N * h.$$

Let $\mathcal{Q} = \{(Q' : \mathfrak{g}) \mid Q' \in \mathcal{Q}'\}$, a set of $\tilde{\mathfrak{g}}$-primes of T. Obviously $|\mathcal{Q}| \leqq p^i$. Since $Q \in \mathcal{Q}$ is $\mathfrak{g}$-invariant, it follows that $Q \cap S = 0$ because S is $\mathfrak{g}$-prime. Also notice that $(\cap \mathcal{Q})^{p^i}$ is $\tilde{\mathfrak{g}}$-invariant and contained in $N * \mathfrak{i}$; therefore,

$$(\cap \mathcal{Q})^{p^i} \subset ((N * \mathfrak{i}) : \tilde{\mathfrak{g}}) \subset (N : \mathfrak{g}) * \mathfrak{i} = 0.$$

Proceeding to primes of $S * \mathfrak{g} = T * \tilde{\mathfrak{g}}$, induction now yields a set of primes $\mathcal{P}_Q$ of $S * \mathfrak{g}$ (for each $Q \in \mathcal{Q}$) satisfying

$$P \cap T = Q, \quad |\mathcal{P}_Q| \leqq p^j, \quad \text{and} \quad (\cap \mathcal{P}_Q)^{p^j} \subset Q * \tilde{\mathfrak{g}},$$

and $\mathcal{P}_Q$ is the set of primes of $S * \mathfrak{g}$ minimal over $Q * \tilde{\mathfrak{g}}$.

Let $\mathcal{P} = \bigcup_{Q \in \mathcal{Q}} \mathcal{P}_Q$, a set of primes of $S * \mathfrak{g}$. First observe that for $P \in \mathcal{P}_Q$, $P \cap S = (P \cap T) \cap S = Q \cap S = 0$. Secondly, observe that

$$|\mathcal{P}| \leqq \sum_{Q \in \mathcal{Q}} |\mathcal{P}_Q| \leqq |\mathcal{Q}| \cdot p^j$$

$$\leqq p^i p^j = p^n.$$

Finally,

$$(\cap \mathcal{P})^{p^n} = \Big(\bigcap_{Q \in \mathcal{Q}} (\cap \mathcal{P}_Q) \Big)^{p^n} \subset \Big(\bigcap_{Q \in \mathcal{Q}} (\cap \mathcal{P}_Q)^{p^j} \Big)^{p^i}$$

$$\subset (\cap Q * \tilde{\mathfrak{g}})^{p^i} = ((\cap \mathcal{Q}) * \mathfrak{g})^{p^i}$$

$$\subset (\cap \mathcal{Q})^{p^i} * \mathfrak{g}.$$

But we saw above that $(\cap \mathcal{Q})^{p^i} = 0$, so, in fact, $(\cap \mathcal{P})^{p^n} = 0$. Thus $\mathcal{P}$ is precisely the set of minimal primes of $S * \mathfrak{g}$ and satisfies the conclusions of the theorem.

Corollary 22. *Let $R * \mathfrak{g}$ be given with $\mathfrak{g}$ finite dimensional abelian. Let $A_1 \subset A_2$ be $\mathfrak{g}$-primes of R, and suppose P_2 is a prime ideal of $R * \mathfrak{g}$ with $P_2 \cap R = A_2$. Then there exists a prime ideal P_1 of $R * \mathfrak{g}$ such that $P_1 \cap R = A_1$.*

Proof. We may assume that $A_1 = 0$ so that R is $\mathfrak{g}$-prime. The theorem says that a minimal prime P_1 contained in P_2 satisfies $P_1 \cap R = 0 = A_1$. □

Prime and primitive rank

A ring R is said to have *prime rank n* if n is the smallest integer such that R has no chain of prime ideals of length greater than n. *Primitive rank* is defined by replacing "prime" with "primitive". If no such bound exists, R is said to have infinite rank. The next theorem shows that if $\mathfrak{g}$ is finite dimensional abelian, the ranks of R and $R * \mathfrak{g}$ agree.

The following lemma shows how primitive ideals of R and $R * \mathfrak{g}$ are related. If A is an ideal of R, let $\hat{A}$ denote the ideal $(A : \mathfrak{g}) * \mathfrak{g}$ of $R * \mathfrak{g}$. *The minimal primes over* $\hat{A}$ are the prime ideals of $R * \mathfrak{g}$, minimal subject to containing $\hat{A}$.

LEMMA 23. *Assume that* $\mathfrak{g}$ *is finite dimensional.*

(i) *Let* P *be primitive ideal of* $R * \mathfrak{g}$. *Then* $N(P \cap R)$ *is a primitive ideal of* R.

(ii) *Let* Q *be a primitive ideal of* R. *Then the minimal primes over* $\hat{Q}$ *in* $R * \mathfrak{g}$ *are all primitive.*

PROOF. (i) Let P be a primitive idealof $R * \mathfrak{g}$ which is the annihilator of the irreducible right $R * \mathfrak{g}$ module V. Of course V is a cyclic $R * \mathfrak{g}$-module, so it is finitely generated as an R module.

By Zorn's Lemma, V has a maximal proper right R-submodule M. Set $W = V/M$, and $Q = \mathrm{ann}_R W$.

We show that $N(P \cap R) = Q$. Clearly $\mathrm{ann}_R V = P \cap R \subset Q$. Thus, since Q is prime, $N(P \cap R) \subset Q$. Conversely, notice that $V(Q : \mathfrak{g}) \subset VQ \subset M$. Thus

$$V \cdot \hat{Q} = V(R * \mathfrak{g})(Q : \mathfrak{g}) = V(Q : \mathfrak{g}) \subset M.$$

But V is an irreducible $R * \mathfrak{g}$-module so $V \cdot \hat{Q} = 0$. This implies that $(Q : \mathfrak{g}) \subset P \cap R$, so $Q^m \subset P \cap R$ for some m by Lemma 1. Thus $Q \subset N(P \cap R)$.

(ii) Let W be an irreducible right R-module with annihilator A. Let $V = W \otimes_R R * \mathfrak{g}$ be the usual induced module.

By considering equations of the form $(W \otimes \bar{x}^j)\alpha = 0$ $(\alpha \in R * \mathfrak{g})$ we deduce that $\mathrm{ann}_{R * \mathfrak{g}} V = \hat{A}$.

Define submodules V_i of V by $V_i = \Sigma_{|\nu| \leq i} W \otimes \bar{x}^\nu$ for all $i \leqq m = (p-1)\dim \mathfrak{g}$. Then $W = V_0 \subset V_1 \subset \cdots \subset V_m = V$, and this is a chain of R submodules of V. Define $\bar{V}_i = V_i / V_{i-1}$. Then $\bar{V}_i$ is a finite sum of copies of W. It follows that V has finite composition length as an R-module, and from this it is immediate that V has a composition series as an $R * \mathfrak{g}$-module. Denote the annihilators of the composition factors of $V_{R * \mathfrak{g}}$ by $P_1, \ldots, P_n$. Now $V \cdot P_1 \cdots P_n = 0$, so $P_1 \cdots P_n \subset \mathrm{ann}_{R * \mathfrak{g}} V = \hat{A}$. Thus any minimal prime over $\hat{A}$ is equal to P_i for some i.

THEOREM 24. *Let* $R * \mathfrak{g}$ *be given with* $\mathfrak{g}$ *finite dimensional abelian. Then the prime (primitive) rank of* $R * \mathfrak{g}$ *is equal to the prime (primitive) rank of* R.

PROOF. Let $P_0 \subsetneqq \cdots \subsetneqq P_n$ be a chain of prime ideals of $R * \mathfrak{g}$. Then, for each i, $Q_i = N(P_i \cap R)$ is the unique minimal prime over $P_i \cap R$ by Lemma 1. Thus

$$Q_0 \subsetneqq \cdots \subsetneqq Q_n$$

is a chain of prime ideals of R, where the inclusions are strict by Theorem 19 and Lemma 18(ii) (with $i = 0$).

Conversely, suppose $Q_0 \subsetneqq \cdots \subsetneqq Q_n$ is a chain of primes of R. Applying Lemma 1, we obtain $Q_0' \subsetneqq \cdots \subsetneqq Q_n'$, a chain of $\mathfrak{g}$-primes of R with $Q_i' = (Q_i : \mathfrak{g})$. By Lemma 4 there is a prime ideal P_n of $R * \mathfrak{g}$ with $P_n \cap R = Q_n'$. Successively applying Going Down (Corollary 22) now yields a chain of primes of $R * \mathfrak{g}$,

$$P_0 \subsetneqq \cdots \subsetneqq P_n,$$

such that $P_i \cap R = Q_i'$ and P_i is a minimal prime over $\hat{Q}_i = Q_i' * \mathfrak{g}$.

Thus we have shown that a chain of primes at length n in $R * \mathfrak{g}$ gives rise to such a chain in R, and vice-versa. To prove the part about primitive rank, we invoke Lemma 23 and replace "prime" with "primitive" throughout the above.

The results above on prime and primitive ranks can be used to study these ranks in the subring of constants $R^{\mathfrak{g}}$ as is done for crossed products in [12]; however, it appears that this would require the assumption that $u(\mathfrak{g})$ is semisimple (analogous to the assumption $|G|^{-1} \in R$ for crossed products $R * G$). This implies that $\mathfrak{g}$ is abelian [8], and after extending the field of scalars, $u(\mathfrak{g}) \cong (kG)^*$ for some finite group G (see [1]). Thus this situation is somewhat special and can be studied in the context of group-graded rings, as is done for certain questions in [1].

An example

Work on finite normalizing extensions [5, 6] and intermediate normalizing extensions [7] yield generalizations of results for crossed products of finite groups [12]. We present an example, pointed out by D. S. Passman, to show that restricted differential operator rings are not covered by this material on normalizing extensions.

Let R be a local ring will radical J. Let $S = \Sigma Rx_i$ be a finite normalizing extension of R. Consideration of the map $R \to Rx_i$ shows that Jx_i is the unique maximal left submodule of Jx_i. Since $Rx_i = x_iR$, we see that x_iJ is a left submodule of x_iR. Hence $x_iJ \subset Jx_i$, and by symmetry $x_iJ = Jx_i$. Thus we have $SJ = JS$.

Further suppose that R is a local k-algebra (char $p > 0$) with a derivation δ satisfying $\delta^p = 0$ and $\delta(r) = 1$ for some $r \in J$. Then we may form the restricted differential operator ring $R * ky$, where $[y, r] = \delta(r)$, $r \in R$, and $y^p = 0$. It is a simple matter to construct such rings R; for example let $R = k[t \mid t^p = 0]$ and let $\delta = d/dt$.

Suppose that $R \subset R * ky \subset S$ where S is a finite normalizing extension of R. Then $JS = SJ$ as above. Now $1 \in \delta(J) = [y, J] \subset SJ$, so $JS = S$. But this contradicts Nakayama's Lemma, since S is a finitely generated R-module. Thus $R * ky$ is not an (intermediate) normalizing extension.

In closing we remark that Theorem 17 yields information on Hopf algebra smash products $R \# H$ where R is a centrally closed prime ring and the action of H is inner on R. Here $R \# H = R_t[H] = R \otimes C_t[H]$ where $R_t[H]$ and $C_t[H]$ are twisted smash products with trivial actions, and $C_t(H)$ centralizes R [3].

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